Exact Algorithms for the 2-Dimensional Strip Packing Problem with and without Rotations

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Abstract. We propose exact algorithms for the 2-dimensional strip packing problem (2SP) with and without 90 degrees rotations. We first focus on the perfect packing problem (PP), which is a special case of 2SP, wherein all given rectangles are required to be packed without wasted space, and design branch-and-bound algorithms introducing several branching rules and bounding operations. A combination of these rules yields an algorithm that is especially efficient for feasible instances of PP. We then propose several methods of applying the PP algorithms to 2SP. Our algorithms succeed in efficiently solving benchmark instances of PP with up to 500 rectangles and those of 2SP with up to 200 rectangles. They are often faster than existing exact algorithms specially tailored for problems without rotations.

1 Introduction

We consider the *2-dimensional strip packing problem* (2SP) that requires all given rectangles to be placed orthogonally without overlap into one rectangular container, called the *strip*, with a fixed width and variable height so as to minimize the height of the strip. The problem is known under various names including the *(orthogonal) rectangular strip packing problem.* Among the many variants of rectangle packing problems, 2SP is one of the problems most intensively studied [8]. As a special case of 2SP, we also consider the *perfect packing problem* (PP) that

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requires a determination of whether all given rectangles can be placed, without overlap or wasted space, into a container with a fixed width and height. Regarding *rotations* of rectangles, we consider the following two cases: (1) rotations of 90 degrees are allowed, and (2) no rotations are allowed. Most of the variants of the rectangle packing problem, including 2SP and PP, are known to be NP-hard. Rectangle packing problems have many applications in the steel and textile industries, and they also have indirect applications in scheduling problems [26] and in other areas [8, 11]. Some theoretical aspects and heuristic/exact algorithms are summarized in [20].

Heuristics and metaheuristics are important in designing practical algorithms for rectangle packing problems. Baker et al. [2] proposed a construction method called the bottom-left-fill (BLF) algorithm for 2SP, and many related papers have appeared; e.g., an efficient implementation [5] and variants of BLF [15, 19]. These algorithms first decide a sequence of rectangles and then place rectangles one by one in this order at an appropriate position. Different types of construction heuristics have also been proposed recently; e.g., the best-fit heuristic [4], a recursive heuristic [31] and others [9, 28]. Dowsland [6] was one of the early papers that presented metaheuristics for rectangle packing problems. Her simulated annealing explores placements directly, allowing infeasible solutions. The genetic algorithm in [3] also works on placements directly. Construction heuristics are often incorporated in metaheuristics to improve the quality of solutions. One of the common ways is to use metaheuristics for searching sequences from which BLF (or similar heuristics) generates good placements [7, 14, 15, 19, 29]. In this scheme, a sequence is an encoded solution and the heuristic such as BLF is a decoding algorithm. Lesh et al. [18] presented a simple algorithm called BLD$*$ for randomly generating good sequences for BLF. Metaheuristics incorporating different types of heuristics have also been proposed; e.g., GRASP [1], local search [28] and genetic algorithm [30]. Murata et al. [25] proposed a simulated annealing and Imahori et al. [11, 12] presented iterated local search algorithms based on a different coding scheme called the sequence-pair representation. For more about heuristic algorithms, see e.g., [13].

With the exception of few papers published recently [17, 21], most results presented in the literature on 2SP and PP concern heuristic algorithms. This may be due to the fact that relative to heuristic algorithms, the size of instances that exact algorithms can handle tends to be small. However, there are many important applications that can be handled by exact algorithms, even with the limitation of a small number of rectangles. Martello et al. [21] proposed exact algorithms for 2SP without rotations and succeeded in solving benchmark instances with up to 200 rectangles. Lesh et al. [17] focused on PP without rotations and solved instances with up to 29 rectangles. The algorithms in these papers exploit the constraint that the rectangles cannot be rotated, and therefore they are not directly applicable to cases where rotations are allowed.

In this paper, we propose exact branch-and-bound algorithms for PP and 2SP *with* and *without* rotations of 90 degrees. We first develop branch-and-bound algorithms for PP. We examine two branching operations, one based on the *bottom-left* point [2] and the other based on a strategy called the *staircase* placement, and we propose new bounding operations based on dynamic programming (DP), linear programming (LP) and others. To solve 2SP using the algorithms for PP, we propose a reduction from 2SP to PP and generalizations of the algorithms for PP.

We confirmed through computational experiments that our algorithm based on the staircase placement and DP bound can solve most of the benchmark instances of PP with up to 49 rectangles and several instances with up to 500 rectangles. For the case without rotations, this algorithm is faster than the one proposed in Lesh et al. [17]. Computational results on benchmark instances of 2SP with up to 200 rectangles reveal that our algorithm based on the staircase placement and another based on the generalization of that algorithm are highly efficient. For the case without rotations, these algorithms are competitive with the method proposed by Martello et al. [21]. Considering the fact that those existing algorithms are specially tailored for the case without rotations, these results are quite satisfactory.

2 Formulations

This section defines the 2-dimensional strip packing problem (2SP) and the perfect packing problem (PP). Though we deal with these problems both with and without rotations of 90 degrees, for conciseness, here we formulate only the problems without rotations. The problems with rotations can be formulated similarly.

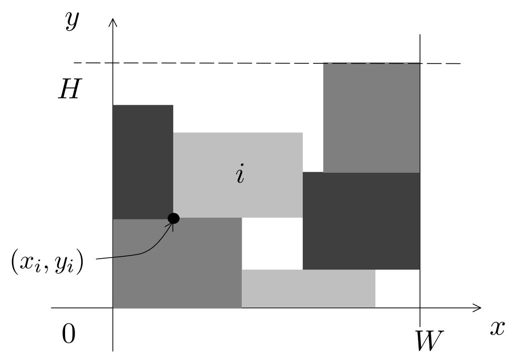


Figure 1. An example of the strip packing problem

2.1 The strip packing problem

Let $I = \{1, 2, \ldots, n\}$ be a set of n rectangles. We are given a width w_i and a height h_i for each rectangle $i \in I$ and a width W for a strip. The strip packing problem requires that n rectangles be placed without overlap into the strip so as to minimize the height H of the strip. We designate the bottom left corner of the strip as the origin of the xy-plane, letting the x-axis be the direction of the width of the strip, and the y-axis be the direction of the height. We represent the location of each rectangle i in the strip by the coordinate (x_i, y_i) of its bottom left corner (see Fig. 1). The set of coordinates $\pi = \{(x_i, y_i) \mid i \in I\}$ is called a *placement* of I. The strip packing problem is formulated as follows:

$$\begin{aligned}
&\text{minimize} && H && && (2.1)\\
&\text{subject to} && x_i + w_i \le W, && \forall i \in I && (2.2)\\
& && y_i + h_i \le H, && \forall i \in I && (2.3)\\
& && x_i + w_i \le x_j \ \text{ or } \ x_j + w_j \le x_i \ \text{ or} && && \\
& && \quad y_i + h_i \le y_j \ \text{ or } \ y_j + h_j \le y_i, && \forall i, j \in I,\ i \ne j && (2.4)\\
& && x_i, y_i \ge 0, && \forall i \in I. && (2.5)
\end{aligned}$$

The constraints (2.2), (2.3) and (2.5) require that all rectangles are within the strip designated with width W and height H. The constraint (2.4) prevents rectangles from overlapping. A placement π is *feasible* if it satisfies the above constraints and is *infeasible* otherwise. According to the improved typology of [27], 2SP is categorized into the two-dimensional regular open dimension problem (2D regular ODP) with a single variable dimension. In this paper, all widths and heights of the rectangles and of the strip are assumed to be integers.

2.2 The perfect packing problem

Given a width w_i and a height h_i for each rectangle $i \in I$ and a width W and a height H of a container, the perfect packing problem requires a determination of whether all given n rectangles can be placed into the container without overlap or wasted space, and if the answer to this determination is yes to then output a feasible solution. An instance of PP is called *feasible* (resp., *infeasible*) if its answer is yes (resp., no). The equation $\sum_{i \in I} w_i h_i = WH$ must hold for the given instance to be feasible; hence this is assumed for PP throughout the remainder of this paper unless otherwise stated. The perfect packing problem is also considered to be a problem of judging whether the optimal value H for the strip packing problem is equal to an obvious lower bound $\sum_{i \in I} w_i h_i / W$.

3 Branch-and-bound method

The branch-and-bound method is one of the representative methodologies for designing exact algorithms for combinatorial optimization problems [10, 17, 21]. It is based on the idea that a problem instance can be solved by dividing it into partial problem instances and then solving all of those. The operation of dividing a problem instance is called a *branching operation*.

We denote the original problem instance by P_0 and the kth partial problem instance generated during computation by P_k. For example, a partial problem for 2SP is defined by a partial solution $\pi_{I'}$, wherein $\pi_{I'}$ is a set of coordinates $\pi_{I'} = \{(x_i, y_i) \mid i \in I'\}$ for a subset $I' \subseteq I$, and it requires the remaining rectangles $I \setminus I'$ to be placed so that the resulting height H is minimized. The process of applying branching operations can be expressed by a rooted tree, called a *search tree*, whose root corresponds to P_0, and the children of a node correspond to the partial problems generated by the branching operation applied to the node; thus each node in the search tree corresponds to a partial problem instance. For 2SP, for example, a branch in the search tree corresponds to fixing the position of a remaining rectangle in $I \setminus I'$. If an optimal solution to P_k is found or if it is concluded that we can obtain an optimal solution to P_0 even without solving P_k or that P_k is infeasible, then it is not necessary to consider P_k further. The operation of removing such a P_k from the list of partial problem instances to be solved is called a *bounding operation*. This operation *terminates* P_k. A partial problem instance is called *active* if it has been neither terminated nor divided into partial problem instances. When no active partial problem instances are left, the entire search terminates and the algorithm delivers an exact optimal solution.

4 Branch-and-bound algorithm for PP

This section describes our branch-and-bound algorithm for PP. This algorithm performs branching operations by fixing absolute positions (coordinates) of rectangles one by one as it goes down the search tree. We describe its basic components and framework in Section 4.1, and then describe the details of branching and bounding rules in Sections 4.2 and 4.3.

4.1 Basic components and framework

The basic components of the algorithm are defined as follows.

Nodes: The root node represents the empty container, and a node of depth d represents a placement $\pi_{I'}$ of a subset $I' \subseteq I$ of d rectangles.

Branching rules: A branch to a child from a node with a placement $\pi_{I'}$ of a subset $I' \subseteq I$ corresponds to placing a rectangle in $I \setminus I'$ at a position in the open space of the current placement $\pi_{I'}$. A branching operation generates those children corresponding to all possible positions of all rectangles in $I \setminus I'$, as well as their possible orientations when the problem with rotations is considered, wherein the candidate positions for placing the rectangles are chosen based on the rules in Section 4.2.1 or 4.2.2.

Bounding rules: Recall that PP is a decision problem. If the algorithm finds that a partial problem does not have a perfect packing, it terminates the corresponding node. The rules for detecting this situation are explained in Section 4.3. If it obtains a perfect packing at a leaf node, then the entire search terminates immediately since in this case the answer is yes.

Search strategy: We adopt the depth first search. The set of all active nodes, denoted by A, is maintained as a stack (an ordered list maintained with the last-in first-out rule); whenever the search moves on to a new active node, it chooses the element most recently added to A. To define the search order among the branches originating from a node, we adopt the following rule.

Prior to our algorithm starting a search, it will sort the given rectangles by non-increasing area (breaking ties by non-increasing width) and will renumber the indices according to this order. When a branching operation generates the children of a node, it adds the generated children to A in decreasing order of indices of the rectangles corresponding to the branches. This means that a branch with a rectangle with a smaller index is searched earlier than those with larger indices. We confirmed through preliminary experiments that this strategy is effective for most instances.

The entire framework of the branch-and-bound algorithm for PP, called algorithm BB-PP, is formally described as follows.

Algorithm BB-PP

Step 0 (initialization). Sort a given set I of rectangles in the order of non-increasing areas (breaking ties by non-increasing width) and renumber the indices according to this order. Let $A := \{\text{the original instance}\}$.

Step 1 (judgement). If A is empty, then output 'infeasible' and stop. Otherwise, let $u \in A$ be the node most recently added to A, and then let $A := A \setminus \{u\}$. Let π_u be the placement corresponding to u. If π_u is a perfect packing of I, then output 'feasible' and stop.

Step 2 (bounding operation). If π_u is a placement of whole I (this means that π_u is infeasible), or if u is terminated by one of the bounding rules in Section 4.3, then return to Step 1. (It is not necessary to use all bounding rules. We examine various combinations in Section 6.)

Step 3 (branching operation). Generate the children of u based on a branching rule in Section 4.2 (BL placement or staircase placement), and add the generated nodes to A in the decreasing order of indices of the placed rectangles. Return to Step 1.

4.2 Branching rules

We consider two branching rules, one based on the bottom left point and the other based on a staircase placement, which will be explained in Sections 4.2.1 and 4.2.2, respectively. Let $B = [0, W] \times [0, H]$ denote the set of all points inside or on the boundary of the container. For a placement $\pi = \{(x_i, y_i) \mid i \in I'\}$ of a subset $I' \subseteq I$, let $C_\pi = \{(x, y) \mid x_i \le x \le x_i + w_i \text{ and } y_i \le y \le y_i + h_i \text{ for some } i \in I'\}$ denote the set of points inside the placed rectangles or on their boundaries. Let $U_\pi = B \setminus C_\pi$, and $\mathrm{cl}(U_\pi)$ denote the closure of U_π, i.e., the minimum closed set including U_π (see Fig. 2(a)).

4.2.1 Branching based on the bottom left point

We first explain the branching based on the bottom left placement for the case without rotations. The leftmost point in the lowest positions of $\mathrm{cl}(U_\pi)$ is called the *BL point* [2]. To be precise, this is the point (x^*, y^*) that satisfies the following two inequalities:

$$y^* \le y, \qquad \text{for all point } (x, y) \in \mathrm{cl}(U_\pi), \tag{4.1}$$

$$x^* \le x, \qquad \text{for all point } (x, y^*) \in \mathrm{cl}(U_\pi). \tag{4.2}$$

For example, Fig. 2(a) shows the BL point (x^*, y^*) of a placement of rectangles. A BL placement is defined as a placement of rectangles obtained by the following rule: Given a permutation of rectangles in I, place rectangles one by one at the BL point in the order of the given permutation. The following lemma is known.

Lemma 1 [17] Every perfect packing is a BL placement for some permutation of the rectangles.

In our branching based on the BL point, each branch corresponds to the act of placing one of the remaining rectangles at the BL point of the current placement; hence each node in the search tree corresponds to the BL placement for some permutation of rectangles corresponding to the path from the root to the node.

For the case of PP with rotations, at each branching operation, the algorithm considers two BL placements (i.e., two branches) for each rectangle, one obtainable by placing it with rotation, and another without rotation.

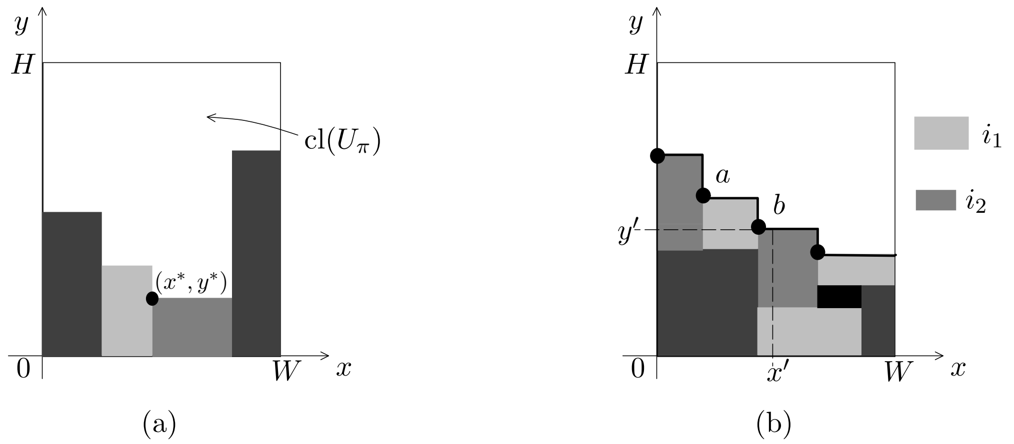


Figure 2. (a)An example of BL point; (b)An example of the staircase placement π with $K(\pi) = 4$

4.2.2 Branching based on the staircase placement

A placement $\pi = \{(x_i, y_i) \mid i \in I'\}$ for a subset $I' \subseteq I$ of rectangles, is called a *staircase placement* if the following two implications hold for arbitrary points $(x, y) \in C_\pi$ and $(x', y') \in U_\pi$:

$$y = y' \Rightarrow x \le x', \tag{4.3}$$

$$x = x' \Rightarrow y \le y' \tag{4.4}$$

(see Fig. 2(b)). In other words, in a staircase placement, the boundary between C_π and U_π forms a monotone (right-down) staircase, and the space below the boundary is filled with rectangles that have been placed without wasted space. The leftmost points of the horizontal lines of the boundary (the four dots in Fig. 2(b)) are called *corner points*. For a staircase placement π, we define the number of the corner points on the staircase as the number of stairs $K(\pi)$.

The algorithm performs branching operations by placing rectangles at corner points, keeping the placement as a staircase placement. Therefore, in a search tree based on this branching operation, a node v is a child of a node u if and only if v corresponds to a staircase placement π_v that is obtainable by placing one of the remaining rectangles at a corner point of the placement π_u of u. It is not difficult to see that the correctness of this branching operation is derived from the following lemma:

Lemma 2 For any staircase placement with at least one rectangle, there exists a rectangle such that the two conditions (4.3) and (4.4) hold even after its removal.

Since the number of stairs $K(\pi)$ is less than or equal to $\lceil n/2 \rceil$ for any staircase placement obtained from a perfect packing of I by Lemma 2, we do not generate any placement such that the number of stairs is more than $\lceil n/2 \rceil$. The idea of using staircase placements was first proposed in [22] in a more general form for 2SP.

In the case of PP with rotations, we consider placements of each rectangle both with and without rotation at each corner point.

Limitation on the number of stairs

Perfect placements sometimes consist of several compound rectangles, and such placements do not require many stairs to place rectangles with the staircase placement strategy. To find such placements more quickly, we introduce the following heuristic rule. We set an upper limit κ on the number of stairs during the search; i.e., if a generated node corresponds to a staircase placement π whose number of stairs is $K(\pi) > \kappa$, then we terminate the node immediately. If a given instance is feasible, a feasible solution is often found more quickly using this heuristic rule [24]. When initiating this heuristic, we first set $\kappa = 2$. If the search terminates all active nodes without finding a feasible placement with the current limit κ, then we increase the limit by one. We repeat this process until κ becomes 4. If we are unable to find a feasible solution with $\kappa = 4$, we increase the limit κ to $\lceil n/2 \rceil$, which is equivalent to the case with no limitation on $K(\pi)$. If no feasible solution is found even with $\kappa = \lceil n/2 \rceil$, then we conclude that the instance is infeasible.

Redundancy check

Branching based on the staircase placement may generate two nodes u, v which correspond to the same placement $\pi_u = \pi_v$. For example, consider two corner points a, b of a placement π for $I' \subseteq I$ and two rectangles $i_1, i_2 \in I \setminus I'$ (see Fig. 2(b)). The node u generated by placing rectangle i_1 at a after placing rectangle i_2 at b and the node v generated by placing rectangle i_2 at b after placing rectangle i_1 at a are descendants of the node for π, and have the same placement $\pi_u = \pi_v$. Below are some techniques we use to reduce such redundancy.

For each rectangle $r \in I$, we introduce a list L_r that stores the positions already searched, and check the list whenever we place rectangle r. This technique is also used in [21], but we have modified it for use with our algorithms as follows: If there are rectangles i and j whose shapes are same (i.e., $w_i = w_j$ and $h_i = h_j$), then they share the same list.

Moreover, we maintain another list that stores all the searched placements with a small number of stairs (to a maximum of two stairs). We check the list whenever the number of stairs $K(\pi_v)$ of the placement π_v of the current node v with rectangle set I' is less than or equal to two. If the list contains a placement π for the same I' and all the coordinates of corner points of π and π_v are the same, then we terminate the node v.

4.3 Bounding operations

This subsection describes three rules for bounding operations: the dynamic programming cut (DP cut), the bounding rule based on the staircase placement and the remaining rectangles, and the LP cut, which are explained in Sections 4.3.1, 4.3.2 and 4.3.3, respectively.

4.3.1 Dynamic programming cut (DP cut)

For a placement π of a subset $I' \subset I$ of rectangles, there are vertical gaps between the top of the container and upper edges of rectangles, and horizontal gaps between the side edges of rectangles or one of the sides of the container (Fig. 3). All such gaps should be filled with the remaining rectangles in $I \setminus I'$ so that a perfect packing for I is obtained. To be more precise, a vertical (resp., horizontal) gap is the length of a vertical (resp., horizontal) line segment that satisfies the following three conditions:

(1) Any point on the line segment is in $\mathrm{cl}(U_\pi)$.

(2) Both end points of the line segment are on the boundary of $\mathrm{cl}(U_\pi)$.

(3) Other points on the line segment are not on the boundary of $\mathrm{cl}(U_\pi)$.

For convenience, we denote that a gap g is *realized* by lengths $l_1, l_2, \ldots, l_k$ if $g = l_1 + l_2 + \cdots + l_k$ holds. If there is a gap that cannot be realized by any combination of the lengths of the remaining rectangles $I \setminus I'$, then we can terminate the node for π. We call this bounding rule *DP cut* and use this with both branching rules in Sections 4.2.1 and 4.2.2.

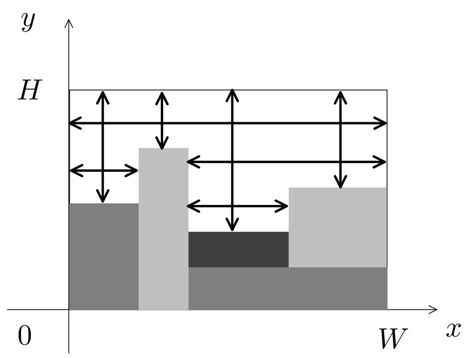


Figure 3. An example of vertical and horizontal gaps for a placement π

We compute whether each of the gaps can be realized by a combination of the lengths, $w_i, h_i, i \in I \setminus I'$, in the following way. We first consider the case with rotations. For a given placement $\pi = \{(x_i, y_i) \mid i \in I'\}$, we denote the remaining rectangles in $I \setminus I'$ by $i = 1, 2, \ldots, m$ for simplicity. The problem of judging whether a given gap can be filled is formulated as a problem similar to the subset sum problem, and it can be solved by using dynamic programming (DP) [23]. Let z_i^{r} (resp., z_i^{u}) be a 0,1-variable that takes value 1 if we use the rotated (resp., unrotated) rectangle i to fill the gap, and 0 otherwise. Then, for $t = 1, 2, \ldots, m$ and an integer $p \geq 0$, the problem $Q_t(p)$ of finding a combination of rectangles from $\{1, 2, \ldots, t\}$ that realizes vertical gap $p \geq 0$ is formally described as follows:

$$
\begin{aligned}
Q_t(p) \quad & \text{Find} && z = (z_1^{\mathrm{r}}, z_2^{\mathrm{r}}, \ldots, z_t^{\mathrm{r}}, z_1^{\mathrm{u}}, z_2^{\mathrm{u}}, \ldots, z_t^{\mathrm{u}}) && (4.5)\\
& \text{such that} && \sum_{i=1}^{t} (w_i z_i^{\mathrm{r}} + h_i z_i^{\mathrm{u}}) = p && (4.6)\\
& && z_i^{\mathrm{r}} + z_i^{\mathrm{u}} \leq 1, \qquad \forall i = 1, 2, \ldots, t && (4.7)\\
& && z_i^{\mathrm{r}}, z_i^{\mathrm{u}} \in \{0, 1\}, \qquad \forall i = 1, 2, \ldots, t. && (4.8)
\end{aligned}
$$

(Note that, though our objective is to find a solution to $Q_m(p)$, for convenience we define the problem for all $t \leq m$.) Let $v_t(p)$ be 1 if there exists a solution to $Q_t(p)$ and 0 otherwise.

Lemma 3 All $v_t(p), t = 1, 2, \ldots, m$ can be computed in $O(m(W + H))$ time.

Proof. We can compute $v_t(p)$ by

$$
v_0(p) = \begin{cases} 1, & p = 0, \\ 0, & \text{otherwise,} \end{cases} \tag{4.9}
$$

$$
v_t(p) = \max\{v_{t-1}(p), v_{t-1}(p - w_t), v_{t-1}(p - h_t)\}, \quad t = 1, 2, \ldots, m, \tag{4.10}
$$

where (4.9) is the boundary condition that represents trivial cases with the empty set of rectangles. Note that by definition $v_t(p) = 0$ holds for any $p < 0$.

Note that a single run of the above algorithm suffices to find the feasibility of all vertical gaps. The same algorithm can also judge the feasibility of horizontal gaps just by exchanging the roles of variables z_i^{r} and z_i^{u}. We can therefore compute the feasibility of all vertical and horizontal gaps by a single run of the above DP recursion. Though we define $v_t(p)$ for all nonnegative integers p, we only need to calculate it for $p = 0, 1, \ldots, \max\{W, H\}$. Hence the above recurrence formula can be calculated in $O(m(W + H))$ time. □

In the case of the problem without rotations, the problem becomes equivalent to the subset sum problem, and we only need to independently execute similar DPs for the horizontal and vertical gaps. When we calculate the DP for the vertical (resp., horizontal) gaps, we remove the second term $v_{t-1}(p - w_t)$ (resp., the third term $v_{t-1}(p - h_t)$) from the recurrence formula (4.10).

A similar idea is also used in [17], where only horizontal gaps are considered and upper bounds on the possible height of the gap are computed based on a slightly more complicated

DP recursion. We simplify their DP, not considering the height but using 0,1-variables, in order to apply it to the case with rotations.

Adaptive control of DP cut

Through preliminary experiments we observed that when the depth of the node in a search tree is relatively small DP cut often fails in terminating nodes. Since executing DP cut is computationally expensive, we incorporate a mechanism to control timing for invoking the DP cut procedure. For this, we use a variable β. We invoke DP cut only if the depth d of the current node satisfies $d \geq \beta$. We control the variable β using two constant parameters δ and q as follows. We set $\beta := 0$ at the root node. Whenever we succeed in terminating a node by a DP cut, we reduce β to $\max\{\beta - \delta, 0\}$. When the DP cut procedure fails in terminating nodes q consecutive times, we increase β by 1. We use $\delta = 1$ and $q = 4$ in the computational experiments in Section 6.

4.3.2 The bounding rule based on the staircase placement and the remaining rectangles

We introduce three simple bounding rules, which are applicable only in branching based on the staircase placement strategy. We terminate the node corresponding to a placement π for I' immediately if one of the following conditions is satisfied: (I) The number of the remaining rectangles $|I \setminus I'|$ is smaller than the number of corner points of π. (II) There is a rectangle r in $I \setminus I'$ which cannot be placed at any corner point of the staircase of π without protruding from the container. (III) All the indices of the remaining rectangles $I \setminus I'$ are smaller than the index of the rectangle placed at the origin $(0, 0)$. The third condition indicates that we have already checked all possible staircase placements formed by $I \setminus I'$ and failed in finding a perfect packing. Hence none of these placements can have the same shape as that of $\mathrm{cl}(U_\pi)$ rotated by 180 degrees.

4.3.3 LP cut

Formulating PP as an integer programming problem, we consider a bounding operation based on its linear programming (LP) relaxation. This operation is applicable to both branching strategies in Sections 4.2.1 and 4.2.2.

We first explain the case with rotations. We define variables $z^{\mathrm{u}}_{i,x,y}$ and $z^{\mathrm{r}}_{i,x,y}$ for $i \in I$ and $(x, y) \in B = \{0, 1, \ldots, W\} \times \{0, 1, \ldots, H\}$, where their meanings are as follows:

- $z^{\mathrm{u}}_{i,x,y} = 1$ if rectangle i is placed at (x, y) without rotation, and 0 otherwise.
- $z^{\mathrm{r}}_{i,x,y} = 1$ if rectangle i is placed at (x, y) with rotation, and 0 otherwise.

Moreover, we define $B^{\mathrm{u}}_i = \{(x, y) \mid x = 0, 1, \ldots, W - w_i, y = 0, 1, \ldots, H - h_i\}$ and $B^{\mathrm{r}}_i = \{(x, y) \mid x = 0, 1, \ldots, W - h_i, y = 0, 1, \ldots, H - w_i\}$ for each $i \in I$. Then PP can be formulated as the following integer program:

$$\text{maximize} \quad \sum_{i\in I}\sum_{(x,y)\in B_i^{\mathrm{u}}} z^{\mathrm{u}}_{i,x,y} + \sum_{i\in I}\sum_{(x,y)\in B_i^{\mathrm{r}}} z^{\mathrm{r}}_{i,x,y} \tag{4.11}$$

$$\text{subject to} \quad \sum_{(x,y)\in B_i^{\mathrm{u}}} z^{\mathrm{u}}_{i,x,y} + \sum_{(x,y)\in B_i^{\mathrm{r}}} z^{\mathrm{r}}_{i,x,y} \le 1, \qquad \forall i \in I \tag{4.12}$$

$$\sum_{i\in I}\sum_{\substack{x-w_i<x'\le x\\ y-h_i<y'\le y}} z^{\mathrm{u}}_{i,x',y'} + \sum_{i\in I}\sum_{\substack{x-h_i<x'\le x\\ y-w_i<y'\le y}} z^{\mathrm{r}}_{i,x',y'} \le 1, \qquad \forall (x,y) \in B \tag{4.13}$$

$$z^{\mathrm{u}}_{i,x,y}, z^{\mathrm{r}}_{i,x,y} \in \{0,1\}, \qquad \forall i \in I, (x,y) \in B \tag{4.14}$$

$$z^{\mathrm{u}}_{i,x,y} = 0, \qquad \forall i \in I, (x,y) \notin B_i^{\mathrm{u}} \tag{4.15}$$

$$z^{\mathrm{r}}_{i,x,y} = 0, \qquad \forall i \in I, (x,y) \notin B_i^{\mathrm{r}}. \tag{4.16}$$

The constraint (4.12) means that each rectangle cannot be placed more than once and (4.13) means that no rectangles can overlap. The original instance for PP has a perfect packing if and only if the optimal value of the corresponding instance of this problem is equal to n.

We now consider the LP relaxation of this problem by relaxing constraints $z^{\mathrm{u}}_{i,x,y}, z^{\mathrm{r}}_{i,x,y} \in \{0,1\}$ to $0 \le z^{\mathrm{u}}_{i,x,y}, z^{\mathrm{r}}_{i,x,y} \le 1$ for all $i \in I$ and $(x,y) \in B$.

Lemma 4 For a given placement $\pi = \{(x_i, y_i) \mid i \in I'\}$ of rectangles, fix the variables $z^{\mathrm{u}}_{i,x,y}, z^{\mathrm{r}}_{i,x,y}$ for the placed rectangles $i \in I'$ accordingly. If the optimal value of the resulting LP instance is less than n, then no perfect packing of I is obtained by extending π.

This formulation contains $\Omega(nWH)$ variables and $WH+n$ constraints. Hence this bounding rule is not practical for problem instances with relatively large WH.

We can also utilize an LP relaxation for the case without rotations. In this case, variables $z^{\mathrm{r}}_{i,x,y}$ are fixed at 0, and therefore are not necessarily introduced in this formulation.

5 Application of PP algorithms to 2SP

This section gives some ideas on applying the algorithms from the previous section to 2SP. We first explain the reduction from 2SP to PP in Section 5.1 and then introduce generalizations of staircase placement, DP cut, and the bounding rule based on the staircase placement and the remaining rectangles in Sections 5.2, 5.3 and 5.4, respectively. We then propose two algorithms for 2SP in Section 5.5.

5.1 Reduction from 2SP to PP

For a given 2SP instance with a set I of rectangles, let H' be an integer such that $WH' \ge \sum_{i\in I} w_i h_i$. The optimal value of the 2SP instance is less than or equal to H' if and only if the following PP instance is feasible. We set the height of the container to H' and add $WH' - \sum_i w_i h_i$ new 1×1 rectangles to the set I so that equation $\sum w_i h_i = WH'$ holds for the resulting instance, where "1×1" means that its height and width are 1.

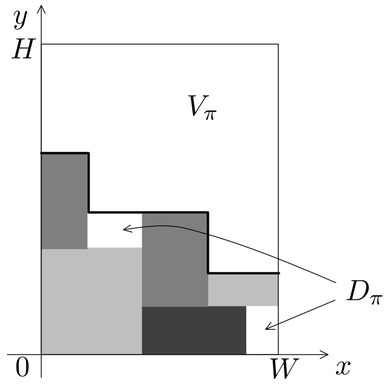


Figure 4. Generalized definition of staircase

Given our assumption that all widths and heights of rectangles are integers, the optimal value of 2SP is also an integer. Hence, 2SP is equivalent to determining the minimum H' such that the corresponding PP instance has a perfect packing.

5.2 Generalizations of staircase placement

Adding too many 1×1 rectangles could lead to a redundant search. We therefore consider an additional idea to generalize the staircase placement. The space below the boundary of a staircase placement must be filled with rectangles placed without wasted space in the original definition in Section 4.2.2. Here we generalize the definition so that it allows wasted space in the space below the boundary. For a given placement π of I', let $V_\pi := \{(x', y') \in U_\pi \mid \text{both (4.3) and (4.4) holds for arbitrary } (x, y) \in C_\pi\}$ and we call $\mathrm{cl}(D_\pi)$ for $D_\pi := U_\pi \setminus V_\pi$ the *abandoned space*, where C_π and U_π are defined as in Section 4.2. Then (4.3) and (4.4) hold for arbitrary points $(x, y) \in C_\pi \cup D_\pi$ and $(x', y') \in V_\pi$. The boundary $\mathrm{cl}(C_\pi \cup D_\pi) \cap \mathrm{cl}(V_\pi)$ becomes a right-down staircase by this definition as shown in thick lines in Fig. 4, where the gray area is the placed rectangles of I'. As in the original staircase, the number of stairs $K(\pi)$ is the number of corner points of the generalized staircase.

As in the branching operation based on the original staircase placement, the algorithm does not put any remaining rectangle into the space below the boundary in the descendants of the search tree; it puts them only at corner points of the staircase. The validity of this branching operation is proved in [22].

5.3 Extension of DP cut

DP cut can be applied directly to 2SP if we adopt the strategy introduced in Section 5.1; i.e., if we calculate DP after adding 1×1 rectangles. However, as the number of 1×1 rectangles increases, gaps are more easily realized and the DP cut becomes less effective. To alleviate this, we propose a new idea of executing the DP computation without 1×1 rectangles explicitly. For a given placement π of I', let g_k^{v} (resp., g_k^{h}) be the length of the kth shortest vertical gap (resp., the kth longest horizontal gap g_k^{h}) in π ($k = 1, 2, \ldots, K(\pi)$), and s_k^{h} (resp., s_k^{v}) be the width (resp., height) of the stair which forms the boundary of the gap in π (see Fig. 5(a)). Let J be a subset of $I \setminus I'$. We then compute l_k^{v} (resp., l_k^{h}), the maximum realizable length by some combination of the lengths of the rectangles in J less than or equal to g_k^{v} (resp., g_k^{h}). Then $s_k^{\mathrm{h}}(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})$ (resp., $s_k^{\mathrm{v}}(g_k^{\mathrm{h}} - l_k^{\mathrm{h}})$) gives a lower bound on the area in V_π that cannot be filled with any combination of rectangles in J. When we sum such areas $s_k^{\mathrm{h}}(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})$ over all vertical gaps and $s_k^{\mathrm{v}}(g_k^{\mathrm{h}} - l_k^{\mathrm{h}})$ over all horizontal gaps, we need to subtract the common area $(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_k^{\mathrm{h}} - l_k^{\mathrm{h}})$ since this space is computed twice. Moreover, $(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_{k+1}^{\mathrm{h}} - l_{k+1}^{\mathrm{h}}),\ k = 1, 2, \ldots, K(\pi) - 1$ gives another area that cannot be filled with rectangles in J. We show that the total of the area, depicted in dark gray in Fig. 5(b), is a lower bound on the area in V_π that cannot be filled with any combination of the rectangles in J.

Lemma 5 For a staircase placement π of I' and a subset $J \subseteq I \setminus I'$ of the remaining rectangles, let g_k^{v} (resp., g_k^{h}) be the length of the kth shortest vertical gap (resp., the kth longest horizontal gap), and let l_k^{v} (resp., l_k^{h}) be the maximum realizable length less than or equal to g_k^{v} (resp., g_k^{h}). Then

$$a(\pi, J) = \sum_{k=1}^{K(\pi)} \{s_k^{\mathrm{h}}(g_k^{\mathrm{v}} - l_k^{\mathrm{v}}) + s_k^{\mathrm{v}}(g_k^{\mathrm{h}} - l_k^{\mathrm{h}}) - (g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_k^{\mathrm{h}} - l_k^{\mathrm{h}})\} + \sum_{k=1}^{K(\pi)-1} \{(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_{k+1}^{\mathrm{h}} - l_{k+1}^{\mathrm{h}})\} \quad (5.1)$$

gives a lower bound on the area in V_π that cannot be filled with any combination of the rectangles in J.

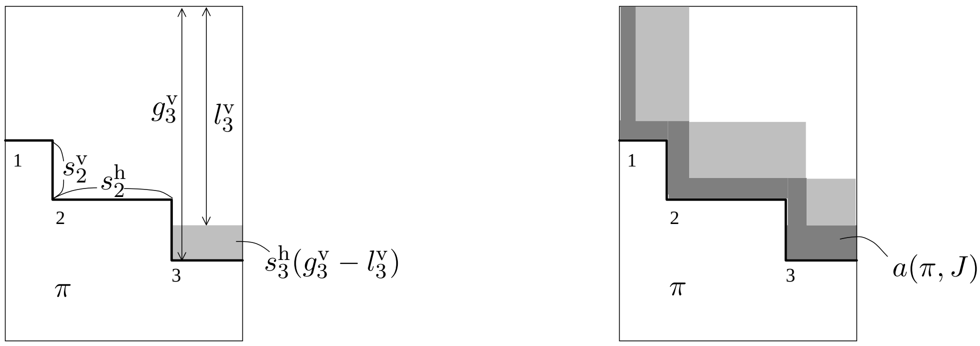
Proof. Consider a staircase placement π and any feasible placement consisting of a set $J' \subseteq J$ of the rectangles placed in V_π. Without loss of generality, we assume that all the positions of the rectangles of J' are rightmost and uppermost, i.e., none of the rectangles in J' can be translated rightward or upward without overlapping with other rectangles in J' or without protruding from the container. Then no rectangles can overlap with the area of $a(\pi, J)$, the dark gray area of Fig. 5(b), because if there exists such a rectangle, then the length from the top edge (resp., right edge) of the container to the bottom edge (the left edge) of the rectangle, which is now realizable with the rectangles in J, would be longer than the maximum realizable length g_k^{v} (resp., g_k^{h}). The area is computed by $\sum_{k=1}^{K(\pi)} \{s_k^{\mathrm{h}}(g_k^{\mathrm{v}} - l_k^{\mathrm{v}}) + s_k^{\mathrm{v}}(g_k^{\mathrm{h}} - l_k^{\mathrm{h}}) - (g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_k^{\mathrm{h}} - l_k^{\mathrm{h}})\} + \sum_{k=1}^{K(\pi)-1} \{(g_k^{\mathrm{v}} - l_k^{\mathrm{v}})(g_{k+1}^{\mathrm{h}} - l_{k+1}^{\mathrm{h}})\}$. □

Hence if

$$a(\pi, J) - \sum_{i \in I \setminus (I' \cup J)} w_i h_i > WH - \sum_{i \in I} w_i h_i - (\text{the area of } \mathrm{cl}(D_\pi)), \quad (5.2)$$

then we can terminate the node. We call this bounding operation *extended DP cut.*

Now, let us turn to the computation of the maximum realizable length l_k^{v} (resp., l_k^{h}). Let the remaining rectangles except for the added 1×1 rectangles be $i = 1, 2, \ldots, m$ for simplicity. We consider $J_1 = \{1\}, J_2 = \{1, 2\}, \ldots, J_m = \{1, 2, \ldots, m\}$ as the candidates of subset J. Then we have the following lemma.



(a) (b)

Figure 5. (a)The definitions of $s_k^{\rm h}, g_k^{\rm v}, l_k^{\rm v}$; (b)An example of $a(\pi, J)$ that cannot be filled

Lemma 6 For the subset J_t with a fixed $t \in \{1, 2, \ldots, m\}$, the maximum realizable lengths $l_k^{\rm v}$ (resp., $l_k^{\rm h}$) for all $k = 1, 2, \ldots, K(\pi)$ are computed in $O(W + H)$ time if $v_t(p)$ of (4.10) are available for all $p = 0, 1, \ldots, \max\{W, H\}$.

Proof. The maximum realizable length $l_k^{\rm v}$ (resp., $l_k^{\rm h}$) is computed as

$$l_k^{\rm v} \text{ (resp., } l_k^{\rm h}) = \max_{p=0,1,\ldots,g_k^{\rm v} \text{ (resp., } g_k^{\rm h})} \{p \mid v_t(p) = 1\}, \tag{5.3}$$

in the process of computing the recurrence formula (4.10). The variables $l_k^{\rm v}$ (resp., $l_k^{\rm h}$) need to be updated only if $v_t(p)$ such that $p \leq g_k^{\rm v}$ (resp., $g_k^{\rm h}$) becomes 1 from 0. $\square$

For all $J_1, J_2, \ldots, J_m$, this bounding operation can be calculated in $O(m(W + H))$ time.

5.4 Extension of the bounding rule based on the staircase placement and the remaining rectangles

This section introduces an extension of the bounding rule based on staircase placement and the remaining rectangles described in Section 4.3.2 in order to apply the branching operation based on the generalized staircase placement and the extended DP cut. (I) If the number of remaining rectangles $|I \setminus I'|$ becomes smaller than the number $K(\pi)$ of corner points of π in a node, then we can add another area that cannot be filled with rectangles in J. Such area $b(\pi, J)$ is defined as the sum of $K(\pi) - |I \setminus I'|$ smallest area of $(g_{k+1}^{\rm h} - l_{k+1}^{\rm h} + s_k^{\rm h} - g_k^{\rm h} + l_k^{\rm h})(g_{k-1}^{\rm v} - l_{k-1}^{\rm v} + s_k^{\rm v} - g_k^{\rm v} + l_k^{\rm v})$, where $g_k^{\rm h}, g_k^{\rm v}, l_k^{\rm h}, l_k^{\rm v} = 0$ if $k \leq 0$ or $k > K(\pi)$ (see the light gray area in Fig. 5(b)). Hence if

$$a(\pi, J) + b(\pi, J) - \sum_{i \in I \setminus (I' \cup J)} w_i h_i > WH - \sum_{i \in I} w_i h_i - (\text{the area of } \mathrm{cl}(D_\pi)), \tag{5.4}$$

then we can terminate the node. (Ⅱ) and (Ⅲ) can be applied directly. We call these bounding operations the *extended bounding rule based on the staircase placement and the remaining rectangles*.

5.5 Algorithms

We propose two algorithms for 2SP; one uses branching operations based on the staircase placement in Section 4.2.2 and adds 1×1 rectangles, and the other uses branching operations based on the generalized staircase placement in Section 5.2 without adding 1×1 rectangles explicitly. We call the resulting algorithms STAIRCASE and G-STAIRCASE, respectively. These two algorithms adopt the extended DP cut in Section 5.3 and the extended bounding rule based on the staircase placement and the remaining rectangles in Section 5.4 as their bounding operations, but they do not use the LP cut in Section 4.3.3.

Algorithm STAIRCASE

We first compute a lower bound LB on the optimal height by

$$LB = \min\left\{\left\lceil \frac{\sum_{i\in I} w_i h_i}{W} \right\rceil, \max_{i\in I} h_i\right\} \tag{5.5}$$

when rotations are not allowed, and by

$$LB = \min\left\{\left\lceil \frac{\sum_{i\in I} w_i h_i}{W} \right\rceil, \max_{i\in I} \min\{h_i, w_i\}\right\} \tag{5.6}$$

when rotations are allowed. We then let $H := LB$ and add $WH - \sum_{i\in I} w_i h_i$ 1×1 rectangles to the set I to obtain a PP instance. We then test whether or not the PP instance is feasible by using algorithm BB-PP with branching operations based on the staircase placement in Section 4.2.2. If we find a feasible solution for the PP instance, then we output the placement as an optimal solution and H as the optimal value and stop. Otherwise, we increase H by one and repeat this procedure until a feasible solution is found.

Algorithm G-STAIRCASE

We first compute the same lower bound LB as STAIRCASE and let $H := LB$. We then test whether or not the PP instance is feasible using algorithm BB-PP with branching operations based on the generalized staircase placement in Section 5.2. If we find a feasible solution for the PP instance, then we output the placement and then stop. Otherwise, we increase H by one and repeat the procedure until a feasible solution is found.

6 Computational results

We report the computational results on algorithm BB-PP in Section 4.1 for PP and algorithms STAIRCASE and G-STAIRCASE in Section 5 for 2SP. In the tables in this section, column 'H^*' shows optimal values, column 'time' shows the computation time in seconds needed to solve the problem instances exactly, and column 'nodes' shows the number of search tree nodes generated by the algorithm. The mark 'T.O.' means that the search did not stop within the time limit. We coded the algorithms in the C language and used a PC with a Pentium 4 (3.0GHz) and 1.0GB memory for computational experiments of this section.

Table 1. Data of the instances generated for our experiment

	W	H	i	1	2	3	4	5	6	7	8	9	10	11	12	13
9perfect	13	20	w_i	4	15	6	3	4	2	8	3	10	-	-	-	-
			h_i	15	3	6	12	6	8	2	5	3	-	-	-	-
9nperfect	13	20	w_i	4	15	6	3	3	2	8	3	10	-	-	-	-
			h_i	15	3	6	12	8	8	2	5	3	-	-	-	-
10nperfect	20	20	w_i	5	9	2	3	4	6	4	9	3	2	-	-	-
			h_i	11	8	8	6	9	8	7	8	11	11	-	-	-
11nperfect	20	20	w_i	4	7	10	2	6	3	1	4	4	6	4	-	-
			h_i	13	8	5	8	7	8	12	7	4	12	8	-	-
12nperfect	20	20	w_i	3	5	4	10	7	6	8	4	18	2	3	9	-
			h_i	6	8	5	8	4	3	8	6	3	3	10	2	-
13nperfect	20	20	w_i	7	1	5	9	2	9	5	5	4	3	2	9	2
			h_i	5	8	9	3	16	7	9	3	9	7	7	3	16

6.1 Instances

The instances used for our computational experiments are: (i) the instance sets N1, N2, N3 and N4 generated by Hopper, which were used in [17] and are available at the ESICUP web site,[1] (ii) the instances n1, n2, ..., n12 generated by Burke et al., which were used in [4] and are available from [4], and (iii) the instances ht01–09, beng01–10, cgcut01–03 and ngcut01–12, which were used in [21] and are available at OR-Library.[2] Each of N1, N2, N3 and N4 consists of five instances, and those in N1 are named n1a, n1b, ..., n1e, and those in N2, N3 and N4 are named similarly. Every instance in (i) and (ii) has a perfect packing (i.e., feasible as PP). We supplemented relatively small PP instances in Table 1 generated by us. They are named with the following rules: 'perfect' (resp., 'nperfect') means that the instance is feasible (resp., infeasible) and the number on the left represents the number of rectangles n.

6.2 Computational results for PP

We first conducted preliminary experiments on relatively small PP instances to compare various combinations of branching rules and bounding rules in algorithm BB-PP. We then chose effective combinations of such rules and tested the resulting algorithms on larger instances.

6.2.1 Comparisons of various rules

We examine which branching and bounding rules are effective for BB-PP. The case with rotations is used for this purpose, and the following three combinations are tested:

- Branching based on the BL point and DP cut,
- Branching based on the BL point, DP cut and LP cut,

[1] http://paginas.fe.up.pt/~esicup/

[2] http://people.brunel.ac.uk/~mastjjb/jeb/info.html

Table 2. Preliminary computational results on PP (with rotations of 90 degrees)

branching operations				BL				staircase	
bounding operations				DP		DP+LP		DP	
name	n	W	H	time (s)†	nodes	time (s)†	nodes	time (s)†	nodes
9perfect	9	13	20	0.01	683	4.21	50	0.00	14
9nperfect	9	13	20	0.04	24,317	15.13	17	0.06	15,791
10nperfect	10	20	20	0.73	340,690	2526.73	4250	1.58	384,269
11nperfect	11	20	20	4.07	1,844,853	T.O.	—	17.30	3,436,031
12nperfect	12	20	20	25.12	9,997,794	T.O.	—	59.31	10,934,146
13nperfect	13	20	20	293.99	119,125,123	T.O.	—	445.76	63,921,968
n1a	17	200	200	1.47	44,459	—	M.O.	0.01	86
n1b	17	200	200	93.80	2,805,337	—	M.O.	0.00	109
n1c	17	200	200	22.04	628,005	—	M.O.	0.00	129
n1d	17	200	200	8.31	251,887	—	M.O.	0.01	265
n1e	17	200	200	0.92	27,467	—	M.O.	0.01	93

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds

- Branching based on the staircase placement and DP cut.

LP is solved by GLPK (glpk-4.7[3]). Adaptive control of DP cut is not incorporated in this experiment to clearly observe the effectiveness of the DP cut.

The results are shown in Table 2. The mark 'M.O.' means that the memory space was not sufficient to solve the LP. As observed in the table, the combination of branching based on the staircase placement and DP cut is very effective for feasible instances, while the combination of branching based on the BL point and DP cut is better for infeasible instances. One conceivable reason for this is that staircase placement limits the number of stairs so that we can find feasible solutions quickly, while this limitation is redundant for infeasible instances. LP cut is not effective since it takes significant time to solve the LP, although it makes the number of nodes very small. Moreover, it consumes a large amount of memory and is not applicable to instances with larger W and H, such as those in set N1. (If available computer memory increases and faster LP solvers become available in the future, then LP cut may become effective.) We therefore restrict our attention to the combinations without LP cut in the subsequent computational experiments.

6.2.2 Computational results for larger instances

The results on instance sets N1, N2, N3 and N4 for problems with and without rotations are reported in Tables 3 and 4, respectively. Throughout this subsection, we denote by 'BL-PP' the algorithm with branching based on the BL point, and by 'StairCasePP' the algorithm based on the staircase placement. DP cut and its adaptive control are incorporated in both cases. The time limit is set to one hour.

[3]http://www.gnu.org/software/glpk/glpk.html

Table 3. Computational results on PP (with rotations of 90 degrees)

				StairCasePP		BL-PP	
name	n	W	H	time (s)†	nodes	time (s)†	nodes
n1a	17	200	200	0.00	86	1.21	44,459
n1b	17	200	200	0.01	109	82.20	3,038,748
n1c	17	200	200	0.00	129	17.54	628,005
n1d	17	200	200	0.00	265	6.69	251,887
n1e	17	200	200	0.00	93	0.70	27,467
n2a	25	200	200	0.08	5074	T.O.	—
n2b	25	200	200	0.59	48,688	T.O.	—
n2c	25	200	200	0.02	1323	T.O.	—
n2d	25	200	200	0.00	321	T.O.	—
n2e	25	200	200	9.38	632,294	T.O.	—
n3a	29	200	200	0.01	823	T.O.	—
n3b	29	200	200	0.35	27,039	T.O.	—
n3c	29	200	200	53.47	3,137,306	T.O.	—
n3d	29	200	200	0.27	31,106	T.O.	—
n3e	29	200	200	0.71	54,826	T.O.	—

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds

Table 4. Computational results on PP (without rotations)

				StairCasePP		BL-PP	
name	n	W	H	time (s)†	nodes	time (s)†	nodes
n2a	25	200	200	0.01	624	3.57	46,072
n2b	25	200	200	0.03	2274	57.15	764,406
n2c	25	200	200	0.01	283	170.17	2,210,939
n2d	25	200	200	0.00	115	45.51	644,484
n2e	25	200	200	0.09	6323	488.85	6,340,166
n3a	29	200	200	0.01	210	T.O.	—
n3b	29	200	200	0.02	1163	T.O.	—
n3c	29	200	200	0.05	6079	536.00	5,951,385
n3d	29	200	200	0.01	438	664.18	7,755,592
n3e	29	200	200	0.02	1380	1526.61	19,009,868
n4a	49	200	200	2743.31	138,563,048	T.O.	—
n4b	49	200	200	280.16	19,305,123	T.O.	—
n4c	49	200	200	T.O.	—	T.O.	—
n4d	49	200	200	2188.01	128,560,407	T.O.	—
n4e	49	200	200	2298.45	125,137,867	T.O.	—

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds

Table 5. Comparison of three algorithms on PP (without rotations)

		StairCasePP†		BL-PP†		Lesh et al.‡	
class	n	avg. time (s)	#solved	avg. time (s)	#solved	avg. time (s)	#solved
N2	25	0.028	5/5	153.05	5/5	≤ 120	5/5
N3	29	0.022	5/5	908.93	3/5	≤ 600	$4/5^{\star}$

†run on a Pentium 4 (3GHz) with the time limit of 3600 seconds
‡run on a Pentium (2GHz) with the time limit of several hours
*All the five instances were solved under a different setting, but the time was not reported.

For the problem with rotations, as shown in Table 3, algorithm StairCasePP solved all instances with $n \leq 29$ within a second except n2e and n3c, and the computation time spent for these two instances were less than a minute. These results indicate that StairCasePP is very efficient. Algorithm BL-PP was able to solve the instances with $n = 17$ in reasonable time, but it is less effective than StairCasePP.

From Table 4, we can observe that StairCasePP is also very efficient for the case without rotations; it was able to solve four out of the five instances with 49 rectangles within the time limit. Algorithm BL-PP was able to solve all instances with $n = 25$ and three out of the five instances with $n = 29$. This is within the time limit, but it is less efficient than StairCasePP as in the case with rotations.

Lesh et al. [17] proposed an exact algorithm for PP without rotations and tested it on these instances with $n \leq 29$. Table 5 summarizes their results and those of StairCasePP and BL-PP, where column "#solved" shows the number of solved instances out of the five, and column "avg. time" shows the average computation time needed to solve an instance.[4] Lesh et al. used a PC with Pentium 2.0GHz. Even considering the difference of the CPUs, StairCasePP seems much faster than their algorithm.

We also tested StairCasePP on the feasible PP instances in [4] for the cases with and without rotations. The results are shown in Table 6. For the case with (resp., without) rotations, our algorithm succeeded in solving nine (resp., eight) out of the twelve instances with up to 500 rectangles within short computation time. Here we note that, while some instances with hundreds of rectangles are efficiently solved, there are some instances for which StairCasePP took long computation time even though the number of rectangles is much smaller (e.g., n4 (resp., n3) for the case with (resp., without) rotations). The characteristics of such hard small instances would be worth investigating.

6.3 Computational results for 2SP

This section reports the computational results of algorithms STAIRCASE and G-STAIRCASE, which were explained in Section 5.5. The results for the case with and without rotations are

[4]The average is taken for those instances solved within the time limit. Lesh et al. reported only approximate computation time (they wrote, e.g., "on average under 10 min") and hence we put "≤"in the table.

Table 6. Computational results of StairCasePP on the instances in [4]

				with rotations of 90 degrees		without rotations	
name	n	W	H	time (s)†	nodes	time (s)†	nodes
n1	10	40	40	0.08	12	0.27	12
n2	20	30	50	0.07	4014	0.08	42
n3	30	30	50	0.47	65,562	289.09	25,633,027
n4	40	80	80	183.61	850,080	23.02	471,555
n5	50	100	100	T.O.	—	T.O.	—
n6	60	50	100	0.15	2348	0.05	95
n7	70	80	100	T.O.	—	0.11	4918
n8	80	100	80	T.O.	—	T.O.	—
n9	100	50	150	0.65	27,049	T.O.	—
n10	200	70	150	0.22	818	0.22	285
n11	300	70	150	4.63	318	0.74	1172
n12	500	100	300	14.90	1138	T.O.	—

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds

shown in Tables 7 and 8, respectively. In the tables, column 'LB' shows the lower bounds by (5.5) or (5.6), and column '1×1' shows the maximum number of 1×1 rectangles added to obtain optimal solutions in STAIRCASE. For comparison purposes, we include in Table 8 as column MMV the results reported in [21]. They conducted their experiments on a PC with a Pentium III (800MHz). The time limit is one hour for STAIRCASE, G-STAIRCASE and MMV. For each row, the shortest computation time is shown in bold face.[5]

For the problem with rotations, as shown in Table 7, both STAIRCASE and G-STAIRCASE were able to solve most of the instances with up to 200 rectangles. The optimal values of the solved instances are equal to or very close to the lower bounds of (5.6). For the problem without rotations, as shown in Table 8, G-STAIRCASE was able to solve most of these instances, while STAIRCASE failed in solving several instances within the time limit. The number of instances solved in the time limit is smaller than the case with rotations. One conceivable reason for this is that the number of instances whose optimal values are equal to the lower bounds of (5.5) is less than that of the case with rotations. The optimal values are the same as the lower bounds for ht and beng instances, while the gap between them is large for about a half of ngcut instances. Because our algorithms are based on PP, they will be more efficient for instances whose optimal values are close to the trivial lower bounds. Indeed, for the ht and beng instance sets, our algorithms solved more instances than MMV. On the other hand, for the ngcut instance set, MMV seems to be more efficient.

[5]We estimated the speed of our CPU (Pentium 4 (3GHz)) to be 3 to 4 times faster than a Pentium III (800MHz). The shortest computation time between those of STAIRCASE and G-STAIRCASE is shown in bold only if it is shorter than that of MMV divided by four except for beng07, for which the results of G-STAIRCASE and MMV are close because $0.56/4 \leq 0.18 \leq 0.56/3$ holds.

Table 7. Computational results on 2SP (with rotations of 90 degrees)

						STAIRCASE		G-STAIRCASE	
name	n	W	LB	H^*	1×1	time (s)†	nodes	time (s)†	nodes
ht01	16	20	20	20	0	0.10	204	**0.07**	204
ht02	17	20	20	20	0	**0.07**	253	0.10	253
ht03	16	20	20	20	0	0.08	26	**0.05**	26
ht04	25	40	15	15	0	**0.10**	2,356	0.12	2,358
ht05	25	40	15	15	0	**0.09**	196	**0.09**	204
ht06	25	40	15	15	0	**0.11**	7,607	**0.11**	7,563
ht07	28	60	30	30	0	**0.09**	36	0.13	36
ht08	29	60	30	30	0	0.19	8,245	**0.14**	8,238
ht09	28	60	30	30	0	**0.09**	484	0.10	484
beng01	20	25	30	30	9	**0.08**	54	**0.08**	55
beng02	40	25	57	57	5	0.12	94	**0.10**	100
beng03	60	25	84	84	10	**0.10**	125	**0.10**	310
beng04	80	25	107	107	2	**0.08**	369	0.13	370
beng05	100	25	134	134	20	**0.08**	185	0.13	198
beng06	40	40	36	36	20	**0.10**	69	0.12	684
beng07	80	40	67	67	7	**0.11**	91	**0.11**	87
beng08	120	40	101	101	13	**0.17**	1,045	0.18	1,027
beng09	160	40	126	126	32	**0.23**	194	0.41	14,332
beng10	200	40	156	156	23	**0.81**	325	3.53	80,505
cgcut01	16	10	23	23	5	0.16	29	**0.10**	107
cgcut02	23	70	63	63	66	**0.18**	13,132	0.75	84,064
cgcut03	62	70	636	—		T.O.	—	T.O.	—
ngcut01	10	10	19	20	10	0.35	3,367	**0.14**	2715
ngcut02	17	10	28	28	3	**0.11**	536	0.14	41
ngcut03	21	10	28	28	3	**0.11**	939	**0.11**	79
ngcut04	7	10	17	18	18	0.29	40,664	**0.15**	581
ngcut05	14	10	36	36	7	**0.08**	2,068	**0.08**	2,390
ngcut06	15	10	29	29	0	0.09	1,632	**0.06**	1,628
ngcut07	8	20	9	10	25	0.39	82,772	**0.13**	1,739
ngcut08	13	20	32	33	27	917.96	72,350,616	**8.80**	946,103
ngcut09	18	20	49	49	6	0.11	2,798	**0.10**	1,779
ngcut10	13	30	58	59	50	T.O.	—	**2.28**	273,329
ngcut11	15	30	50	—		T.O.	—	T.O.	—
ngcut12	22	30	77	77	14	**8.54**	978,092	12.66	935,604

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds

Table 8. Computational results on 2SP (without rotations)

						STAIRCASE		G-STAIRCASE		MMV
name	n	W	LB	H^*	1×1	time (s)†	nodes	time (s)†	nodes	time (s)‡
ht01	16	20	20	20	0	0.10	52	**0.07**	52	10.84
ht02	17	20	20	20	0	0.12	1,325	**0.07**	1,315	623.53
ht03	16	20	20	20	0	**0.08**	22	0.10	22	500.75
ht04	25	40	15	15	0	**0.08**	3,547	0.11	3,545	8.26
ht05	25	40	15	15	0	0.09	599	**0.06**	599	20.29
ht06	25	40	15	15	0	0.11	2,215	**0.06**	2,212	16.94
ht07	28	60	30	30	0	0.12	1,340	**0.10**	1,340	T.O.
ht08	29	60	30	30	0	**71.40**	3,030,967	76.97	2,957,065	T.O.
ht09	28	60	30	30	0	0.10	116	0.13	116	**0.00**
beng01	20	25	30	30	9	**0.53**	74,209	0.93	95,784	511.58
beng02	40	25	57	57	5	**1.13**	94,169	22.89	463,205	T.O.
beng03	60	25	84	84	10	0.94	40,653	**0.32**	10,311	T.O.
beng04	80	25	107	107	2	**0.25**	5,309	T.O.	—	T.O.
beng05	100	25	134	134	20	**0.15**	4,345	0.31	8,530	500.62
beng06	40	40	36	36	20	**0.07**	204	0.29	25,369	T.O.
beng07	80	40	67	67	7	0.33	9,021	**0.18**	802	**0.56**
beng08	120	40	101	101	13	**1.36**	33,596	2.67	77,150	500.54
beng09	160	40	126	126	32	0.42	7,047	2.38	39,467	**0.03**
beng10	200	40	156	156	23	2.86	6,977	6.52	101,671	**0.03**
cgcut01	16	10	23	23	5	**0.10**	3,789	0.12	3,578	11.48
cgcut02	23	70	63	—		T.O.	—	T.O.	—	T.O.
cgcut03	62	70	636	—		T.O.	—	T.O.	—	T.O.
ngcut01	10	10	19	23	40	2080.04	257,058,531	0.39	23,854	**0.05**
ngcut02	17	10	28	30	13	T.O.	—	T.O.	—	**11.31**
ngcut03	21	10	28	28	3	**0.09**	184	0.10	152	27.01
ngcut04	7	10	17	20	38	3.63	766,572	0.14	106	**0.00**
ngcut05	14	10	36	36	7	0.11	56	0.07	83	**0.00**
ngcut06	15	10	29	31	20	T.O.	—	**147.31**	4,423,772	727.20
ngcut07	8	20	20	20	225	T.O.	—	0.10	16	**0.00**
ngcut08	13	20	32	33	27	30.51	3,911,039	**0.50**	39,291	53.09
ngcut09	18	20	49	50	26	T.O.	—	1971.64	42,196,600	T.O.
ngcut10	13	30	58	80	680	T.O.	—	113.98	12,642,065	**0.18**
ngcut11	15	30	50	52	77	T.O.	—	**7.71**	573,883	483.01
ngcut12	22	30	77	87	314	T.O.	—	T.O.	—	0.00

†on a Pentium 4 (3GHz) with the time limit of 3600 seconds
‡on a Pentium III (800MHz) with the time limit of 3600 seconds

7 Branch-and-bound algorithm based on sequence pairs

We examined another branch-and-bound algorithm based on a different scheme for representing solutions. It is based on the *sequence pair* representation proposed in [25], which defines relative positions of rectangles by using a pair of permutations of the set I. An interesting feature of this scheme is that polynomial-time algorithms for encoding and decoding are known [11, 25], where encoding is to find the sequence pair representation that does not contradict a given placement, while decoding is to find the best placement among those that satisfy the constraints on the relative positions of rectangles defined by a given sequence pair.

One of the merits of this branch-and-bound algorithm is that it can handle instances having non-integer values for widths and heights, while the algorithms proposed in Sections 4 and 5 and those in [17, 21] require that all input values are integers.

Our algorithm based on sequence pairs could solve instances with up to 10 (resp., 11) rectangles for the case with (resp., without) rotations within the time limit of two hours on a PC with a Pentium 4 (3GHz). However, these results are not competitive with those reported in Section 6. We therefore omit the details of this algorithm in this paper. The description of this algorithm and its computational results are reported in [16].

8 Conclusions

We proposed several ideas for exactly solving the perfect packing (PP) and strip packing (2SP) problems using the branch-and-bound method. We confirmed through computational experiments that branching based on the staircase placement is effective for PP. Our algorithm based on this idea was able to solve all the benchmark instances with up to 29 rectangles for the case with rotations of 90 degrees and was able to solve all but one instance with up to 49 rectangles for the case without rotations. Moreover, it succeeded in solving several benchmark instances with up to 500 rectangles in less than 15 seconds. We also observed that our algorithms for 2SP were efficient for instances whose optimal values are close to the trivial lower bounds; they solved most of the benchmark instances with up to 200 rectangles within one hour. For the case without rotations, one of our algorithms for PP outperformed the algorithm in [17], and our algorithms for 2SP were competitive with the method in [21]. Considering the fact that the existing algorithms are tailored for the case without rotations, these results are quite satisfactory.

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